

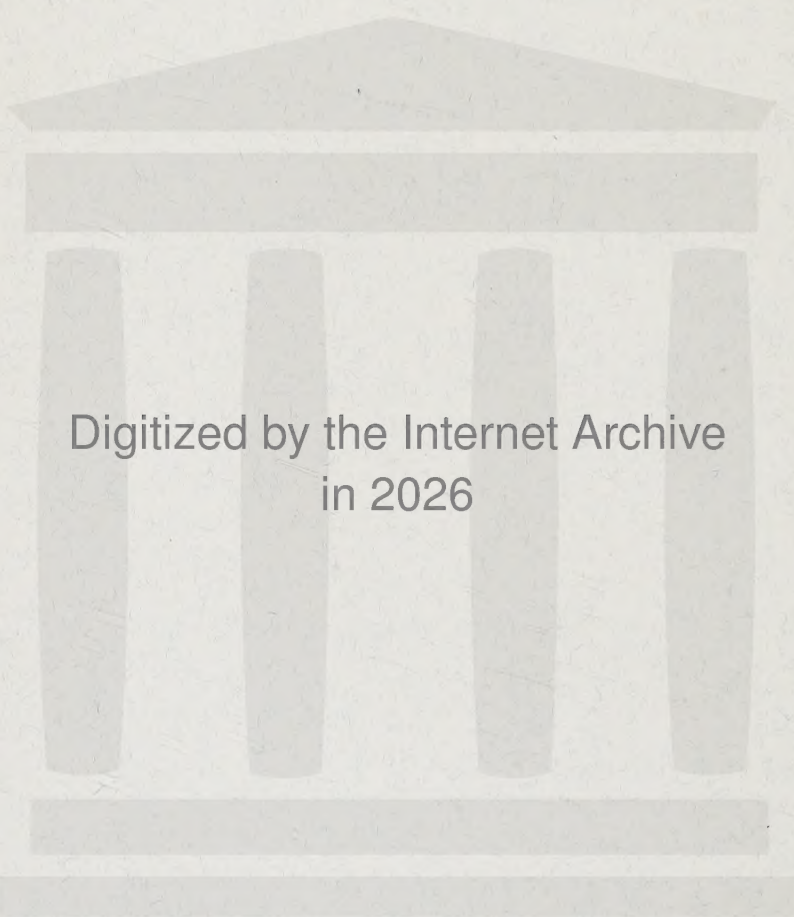
ON TYPES OF "WEAK" CONVERGENCE IN LINEAR NORMED
SPACES

BY

INGO MADDAUS, JR.

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR DEGREE OF DOCTOR OF PHILOSOPHY
IN THE UNIVERSITY OF MICHIGAN

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BY INGO MADDAUS, JR.

(July 28, 1939)

Introduction. In his paper "On a generalized notion of convergence in a Banach space," B. Vulich (Vulich 1, pp. 156-174) has pointed out that it is often possible to introduce a metric into a given concrete space with a given notion of convergence of a sequence of elements such that convergence in accordance with the metric is equivalent to the given convergence notion. However, it is not always possible to do this in terms of a metric which is a distance function (Vulich 1, p. 163). Consequently, Vulich has introduced a generalized metric by associating with each finite set or complex of a given linear set¹ a non-negative real number or norm. A set of six axioms imposed on this generalized metric and a limit notion defined in terms of it give rise to the concept of K -normed spaces.² The K -normed spaces are found to be special cases of Banach spaces (Banach, p. 53), and the K -convergence of a sequence of elements to a limiting element is found to imply Banach convergence to the same element. That is to say, K -convergence of a sequence to an element is stronger than Banach convergence of the sequence to the same element.

In this paper, also, the notion of the norm of a finite complex is employed. However, this norm is subjected to only three of the six axioms used in the definition of the K -normed spaces of Vulich. A limit notion for sequences is defined in terms of this norm. Among other things it is shown that point-wise convergence of a sequence of continuous functions to a continuous function on

¹ By a linear set will be meant the usual one given in Banach's book on page 26.

² A linear set X is said to be K -normed if there is associated with each finite set or complex of its elements a non-negative real number or norm, written $|| (x_1, x_2, \dots, x_n) ||$, which satisfies the following axioms:

Axiom A. If $x_i = x_j$, then $|| (x_1, \dots, x_i, \dots, x_j, \dots, x_n) || = || (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_j, \dots, x_n) || = || (x_1, \dots, x_i, \dots, x_{j-1}, x_{j+1}, \dots, x_n) ||$.

Axiom B. $|| (x) || = 0$ implies $x = \theta$, where θ is the null element of the linear set.

Axiom C. $|| (x_1, \dots, x_n, x'_1, \dots, x'_n) || \leq || (x_1, \dots, x_n) || + || (x'_1, \dots, x'_n) ||$.

Axiom D. $|| (x_1, \dots, x_n) || \leq || (x_1, \dots, x_n, x_{n+1}) ||$.

Axiom E. $|| (x_1 + x'_1, \dots, x_n + x'_n) || \leq || (x_1, \dots, x_n) || + || (x'_1, \dots, x'_n) ||$.

Axiom F. $|| (ax_1, \dots, ax_n) || = |a| \cdot || (x_1, \dots, x_n) ||$.

By $x_n \rightarrow x$ it is meant that given any $\epsilon > 0$ there exists an $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies $|| (x_n - x, \dots, x_{n+p} - x) || < \epsilon$ for all $p \geq 0$. The sequence $\{x_n\}$ is said to be K -convergent if given any $\epsilon > 0$ there exists an $N(\epsilon)$ such that if $n, m \geq N(\epsilon)$ then $|| (x_n - x_m, \dots, x_{n+p} - x_{m+p}) || < \epsilon$ for all $p \geq 0$.

A linear set is said to be a K -normed space if it is K -normed and if the limit notions are given by the definitions of the previous paragraph.

the finite closed interval reduces to the convergence of the sequence to the same function in terms of the norm of this paper. In all concrete cases considered it is seen that convergence of a sequence to an element in accordance with the limit notion to be defined is weaker than Banach convergence of the sequence to the element. Linear functionals and operations which are continuous in terms of this limit notion are studied.

1. Axioms and fundamental notions. Consider a linear set X of elements x such that there is associated with each finite subset or complex of elements of X a non-negative real number, called the norm of the complex and written $\| (x_1, x_2, \dots, x_n) \|$ for the elements $x_1, x_2, \dots, x_n$. This norm will satisfy the following axioms:

AXIOM A. If $x_i = x_j$, then $\| (x_1, \dots, x_i, \dots, x_j, \dots, x_n) \| = \| (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_j, \dots, x_n) \| = \| (x_1, \dots, x_i, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \|$.

AXIOM B. $\| (x) \| = 0$ implies $x = \theta$, where θ is the null element of the linear set X .

AXIOM 3. If $a_1, a_2, \dots, a_n$ are real constants, then $\| (a_1 x_1, a_2 x_2, \dots, a_n x_n) \| \leq \max_{1 \leq i \leq n} |a_i| \cdot \| (x_1, x_2, \dots, x_n) \|$.

A linear set with the norm of a finite complex defined and satisfying Axioms A, B, and 3 will be said to be H -normed.

DEFINITION 1. A sequence $\{x_n\}$ of elements of an H -normed set X will be said to be convergent in the H -sense, or to be H -convergent, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that $k, s \geq K_0$ imply

$$(1) \quad \| (x_{n_1} - x_{n_s}, \dots, x_{n_k} - x_{n_s}) \| < e.$$

DEFINITION 2. A sequence $\{x_n\}$ of elements of an H -normed set X will be said to be convergent to the element $x \in X$ in the H -sense, or to have x as its H -limit, written $x_n \rightarrow_H x$ or $H\text{-}\lim_{n \rightarrow \infty} x_n = x$, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that $k \geq K_0$ implies

$$(2) \quad \| (x_{n_1} - x, \dots, x_{n_k} - x) \| < e.$$

An H -normed set for which the limit notions are given by Definitions 1 and 2 will be referred to as an H -normed space.

It is evident that in an H -normed space if a sequence converges or converges to an element x then every subsequence does likewise.

REMARK 1.1. It should be noted that our Axioms A and B are the Axioms A and B of K -normed spaces and Axiom 3 is a consequence of the axioms of K -normed spaces (Vulich 2, p. 61). Axioms A, B, and 3 are independent (Vulich 2, p. 60), and the examples which will be given in §2-5 will illustrate that there are H -normed spaces which are not K -normed according to the same norm.

The set of all real numbers forms an H -normed space if $\| (x_1, x_2, \dots, x_n) \| = \inf (|x_1|, |x_2|, \dots, |x_n|)$, where by "inf" is meant the smallest of the numbers involved. If in this same set $x_n \rightarrow_B x$ means that given any $\epsilon > 0$ there exists an $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies that $|x_n - x| < \epsilon$, then we have

THEOREM 1.1. *In the set of all real numbers $x_n \rightarrow_H x$ is equivalent to $x_n \rightarrow_B x$.*

PROOF. For any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\| (x_{n_1} - x, \dots, x_{n_k} - x) \| = \inf (|x_{n_1} - x|, \dots, |x_{n_k} - x|) \leq |x_{n_k} - x|$. If $x_n \rightarrow_B x$ then this last expression approaches zero as $k \rightarrow \infty$. To prove the converse assume that $x_n \rightarrow_H x$ does not imply $x_n \rightarrow_B x$. Then there exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a positive number λ such that $|x_{n_k} - x| > \lambda > 0$ for $k = 1, 2, \dots$. Then $0 < \lambda < \| (x_{n_1} - x, \dots, x_{n_k} - x) \|$ for every integral k . This contradicts the assumption that $x_n \rightarrow_H x$.

REMARK 1.2. In general any linear set for which an F -metric (Banach, p. 35) is defined is an H -normed space if $\| (x_1, \dots, x_n) \| = \inf [(x_1, \theta), \dots, (x_n, \theta)]$. By Theorem 1.1 $x_n \rightarrow_H x$ is equivalent to $x_n \rightarrow x$ in accordance with the F -metric. Hence, any linear normed space³ is also an H -normed space, and when the norm of a finite complex is defined as in this paragraph norm convergence of a sequence to an element is equivalent to H -convergence to the same element. Under the same circumstances norm convergence of a sequence is equivalent to H -convergence of the sequence.

The following theorems relate to H -normed spaces.

THEOREM 1.2. *If $x_n \rightarrow_H x$ and $x_n = x'_n$ for every n , then $x'_n \rightarrow_H x$.*

THEOREM 1.3. *If $x_n \rightarrow_H x$, then for any $x' \in X$, $x_n + x' \rightarrow_H x + x'$.*

REMARK 1.3. In Remark 3.1 an example will be given which will show that $x_n \rightarrow_H x$ and $y_n \rightarrow_H y$ do not imply that $x_n + y_n \rightarrow_H x + y$.

Because of Theorem 1.3 it is evident that the topology of an H -normed space is a uniform topology.

THEOREM 1.4. $\| (ax_1, \dots, ax_n) \| = |a| \cdot \| (x_1, \dots, x_n) \|$.

PROOF. $\| (ax_1, \dots, ax_n) \| \leq |a| \cdot \| (x_1, \dots, x_n) \| = |a| \cdot \| (aa^{-1}x_1, \dots, aa^{-1}x_n) \| \leq |a| \cdot |a^{-1}| \cdot \| (x_1, \dots, x_n) \| = \| (ax_1, \dots, ax_n) \|$. This completes the proof.

THEOREM 1.5. *If $\{a_n\}$ is a sequence of real numbers such that $|a_n| \leq M$, where M is independent of n , and if $x_n \rightarrow_H \theta$, then $a_n x_n \rightarrow_H \theta$.*

PROOF. Choose any infinite subsequence $\{a_{n_k} x_{n_k}\}$ of $\{a_n x_n\}$. Then $\| (a_{n_1} x_{n_1}, \dots, a_{n_k} x_{n_k}) \| \leq M \| (x_{n_1}, \dots, x_{n_k}) \|$. Since $x_n \rightarrow_H \theta$ it follows that the right hand side of this inequality approaches zero as $k \rightarrow \infty$.

COROLLARY 1.51. *If $\{a_n\}$ is a sequence of real numbers such that $a_n \rightarrow_B 0$ and if $x_n \rightarrow_H \theta$, then $a_n x_n \rightarrow_H \theta$.*

COROLLARY 1.52. *If a is a real number and $x_n \rightarrow_H x$, then $ax_n \rightarrow_H ax$.*

³ A linear normed space is a linear set X with the property that there is associated with each element x a non-negative real number or norm, designated by $\|x\|$, which is such that (1) $\|x\| = 0$ implies $x = \theta$, (2) $\|x + y\| \leq \|x\| + \|y\|$, (3) $\|tx\| = |t| \cdot \|x\|$ for each real t . A sequence $\{x_n\}$ of X is said to be convergent to $x \in X$ in the norm or Banach sense if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

2. The space (CH). Consider the set of all continuous functions defined on the closed interval $(0, 1)$, and let the norm of a finite complex of elements be given by $\| (x_1, \dots, x_n) \| = \max_{0 \leq t \leq 1} \inf (|x_1(t)|, \dots, |x_n(t)|)$. With this norm the set becomes an H -normed space and it will be designated by (CH) .

THEOREM 2.1. *In the space (CH) a necessary and sufficient condition that $x_n \rightarrow_H x$ is that $x_n(t) \rightarrow x(t)$ for each $t \in (0, 1)$.*

PROOF. Because of Theorem 1.3 it suffices to prove this theorem for the case $x_n \rightarrow_H \theta$.

Necessity. Assume on the contrary that for some t_0 , $\overline{\lim} |x_n(t_0)| = 2\lambda > 0$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with the property that $|x_{n_k}(t_0)| > \lambda$ for each integral k . Therefore, $\| (x_{n_1}, \dots, x_{n_k}) \| \geq \inf (|x_{n_1}(t_0)|, \dots, |x_{n_k}(t_0)|) > \lambda > 0$ for each integral k . Consequently, θ is not an H -limit of $\{x_n\}$, in contradiction to the assumption of the theorem.

Sufficiency. Assume that there is a subsequence of $\{x_n\}$, which will also be designated by $\{x_n\}$, such that $\| (x_1, \dots, x_n) \|$ is greater than a positive number λ for $n = 1, 2, \dots$. Designate $\inf (|x_1(t)|, \dots, |x_n(t)|)$ by $f_n(t)$. For a fixed value of n it is easily shown that $f_n(t)$ is a continuous function on the closed interval $(0, 1)$. Hence, there exists a t_n such that $f_n(t_n) = \max_{0 \leq t \leq 1} |f_n(t)| = \| (x_1, \dots, x_n) \|$. By the Weierstrass-Bolzano Theorem it is possible to choose a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a point t_0 such that $\lim_{k \rightarrow \infty} t_{n_k} = t_0$.

Each $f_n(t_0) \geq \lambda$. For if $f_{n'}(t_0) < \lambda$ there would exist an interval I about t_0 such that if $t \in I$ then $f_{n'}(t) < \lambda$. Now choose $n_k > n'$ and such that $t_{n_k} \in I$; then $f_{n_k}(t_{n_k}) < f_{n'}(t_{n_k}) < \lambda$, which is a contradiction. But $f_n(t) \leq |x_n(t)|$ for all n and t , so $|x_n(t_0)| > f_n(t_0) \geq \lambda$, in contradiction to the assumption that $x_n(t) \rightarrow 0$ for all $t \in (0, 1)$.

REMARK 2.1. If $x_1, x_2, \dots, x_n$ and $x'_1, x'_2, \dots, x'_n$ are two finite complexes of an H -normed space it does not follow that $\| (x_1 + x'_1, x_2 + x'_2, \dots, x_n + x'_n) \| \leq \| (x_1, x_2, \dots, x_n) \| + \| (x'_1, x'_2, \dots, x'_n) \|$. For consider the following complexes from the H -normed space (CH) : $x = 2, x = 5$ and $x' = 5, x' = 2$. In this case the inequality sign is actually reversed. If the sign of x_1 is changed then the inequality sign is again reversed. Therefore, unlike the norm of the K -normed spaces of B. Vulich, the norm of a finite complex of a H -normed space does not satisfy a "triangle property," nor does it satisfy a "triangle property" with the sign reversed.

REMARK 2.2. In the space (CH) it can readily be shown by methods similar to those employed in the proof of Theorem 2.1 that a necessary and sufficient condition for the convergence of a sequence in the H -sense is: $|x_n(t) - x_m(t)| \rightarrow 0$ for each $t \in (0, 1)$. There are numerous familiar examples from the space (CH) which illustrate that H -convergence of a sequence does not imply that the sequence has an H -limit which is of the space. That is to say, an H -normed space is not necessarily complete with respect to the H -convergence notions.

REMARK 2.3. The norm of a finite number of elements of an H -normed space

is not necessarily a continuous function of its arguments; i.e., $x_n^{(i)} \rightarrow_H x^{(i)}$ for $i = 1, 2, \dots, p$ does not imply $\lim_{n \rightarrow \infty} \|(x_n^{(1)}, \dots, x_n^{(p)})\| = \|(x^{(1)}, \dots, x^{(p)})\|$.

For consider the following sequence from (CH):

$$x_n(t) = x_n(0) = 0 \text{ for } 2/n \leq t \leq 1,$$

$$x_n(1/n) = n,$$

and $x_n(t)$ linear elsewhere ($n = 1, 2, \dots$) (Caratheodory, p. 171). This sequence is such that $x_n \rightarrow_H \theta$. Let a second sequence be given by $x'_n(t) = 1$ for each n and for $t \in (0, 1)$. Then $x'_n \rightarrow_H 1$. Moreover, $\|(x_n, x'_n)\| = 1$ for each n , whereas $\|(\theta, 1)\| = 0$.

3. The space $(L^p H)$, $p \geq 1$. The norm of a finite complex which was used in the previous paragraph to make the set of all continuous functions on the finite closed interval an H -normed space was obtained by operating on the greatest lower bound of the absolute values of the elements involved with the norm which is generally employed to make the set of all continuous functions a Banach space (Banach, p. 11). To define $\|(x_1, \dots, x_n)\|$ in an analogous manner for the set of all functions defined on the closed interval $(0, 1)$ and

whose p th powers are summable set it equal to $\left\{ \int_0^1 [\inf(|x_1(t)|, \dots, |x_n(t)|)]^p dt \right\}^{1/p}$, where by "inf" is meant the greatest lower bound of the func-

tions involved at every point of $(0, 1)$.⁴ With this interpretation of the norm of a finite complex the set of functions considered is an H -normed space. This space will be referred to as the space $(L^p H)$, $p \geq 1$. It will be seen from what follows that the H -convergence of a sequence of elements of $(L^p H)$, $p \geq 1$, to an element of the space is extremely weak.

LEMMA 3.1. *If $\{f_n\}$ is a sequence of summable functions defined on $(0, 1)$ and such that*

$$(1) f_n(t) \geq 0 \text{ for each } n,$$

$$(2) \{f_n(t)\} \text{ is non-increasing,}$$

$$(3) \lim_{n \rightarrow \infty} \int_0^1 f_n(t) dt = 0,$$

then $\lim_{n \rightarrow \infty} f_n(t) = 0$ except on a set of zero measure.

THEOREM 3.1. *In $(L^p H)$, $p \geq 1$, $x_n \rightarrow_H x$ is equivalent to either of the following statements:*

(1) *For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\inf_k |x_{n_k}(t) - x(t)| = 0$ except on a set of measure zero which depends on the subsequence.*

⁴ All the work of this section would go through without difficulty if the "inf" should be defined as the greatest lower bound of the functions involved except on a set of zero measure.

(2) For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there is a set E of zero measure such that for $t_0 \notin E$ there is a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\lim_{l \rightarrow \infty} x_{n_{k_l}}(t_0) = x(t_0)$.

PROOF. Because of Theorem 1.3 it is sufficient to prove this theorem for the case $x_n \rightarrow_H \theta$. We first show that $x_n \rightarrow_H \theta$ implies condition (2). Let $f_n^s(t) = [\inf (|x_s(t)|, \dots, |x_n(t)|)]^p$ for each $t \in (0, 1)$. If s is constant then by Lemma 3.1 $\lim_{n \rightarrow \infty} f_n^s(t) = 0$ except on a set E_s of zero measure. Let $E = \sum_{s=1}^{\infty} E_s$; then

E is of zero measure and it is the set mentioned in the statement of the theorem. For consider any t_0 of the complement of E ; $\lim_{n \rightarrow \infty} f_n^s(t_0) = 0$ for each s , and in

particular for $s_0 = 1$. Hence, there exists an integer $s_1 > s_0$ such that $|x_{s_1}(t_0)| < 1$. Consider $\{f_n^{s_1+1}\}$; $\lim_{n \rightarrow \infty} f_n^{s_1+1}(t_0) = 0$, and so there exists an integer $s_2 > s_1$ such that $|x_{s_2}(t_0)| < \frac{1}{2}$. In this way an increasing sequence $\{s_i\}$ of integers can be built up in such a way that $|x_{s_i}(t_0)| < 1/i$. Then $\lim_{i \rightarrow \infty} x_{s_i}(t_0) = 0$.

Therefore, E is the set mentioned in statement (2). Since $x_{n_k} \rightarrow_H \theta$ is a consequence of $x_n \rightarrow_H \theta$, and since we may apply the above argument to $\{x_{n_k}\}$, our contention is proved.

It is obvious that statement (2) implies statement (1).

Now suppose that (1) holds when $x = \theta$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and let $f_k(t) = [\inf (|x_{n_1}(t)|, \dots, |x_{n_k}(t)|)]^p$. Then (1) states that $\inf_k |x_{n_k}(t)| = \lim_{k \rightarrow \infty} [f_k(t)]^{1/p} = 0$ except on a set E of zero measure. Hence,

$\lim_{k \rightarrow \infty} f_k(t) = 0$ except on E and $\lim_{k \rightarrow \infty} \int_0^1 f_k(t) dt = 0$. Therefore, $x_n \rightarrow_H \theta$.

REMARK 3.1. The following example from $(L^p H)$ shows that in an H -normed space the H -limit of a sequence is not necessarily unique. Let $\{x_n\}$ be defined by $x_n(t) = \frac{1}{2}(1 + r_n(t))$, where $r_n(t) = \text{sgn}(\cos 2\pi nt)$.⁵ If $E_n = E(x_n(t) = 1)$, then $\{E_n\}$ is a sequence of sets of closed intervals with no points common to any pair. This sequence has the property that the measure of the points common to each E_{n_k} of any subsequence $\{E_{n_k}\}$ is zero. Consequently, each

$\{x_{n_k}\}$ of $\{x_n\}$ has the property that $\lim_{k \rightarrow \infty} \int_0^1 [\inf (|x_{n_1}(t)|, \dots, |x_{n_k}(t)|)]^p dt = 0$,

and so $x_n \rightarrow_H \theta$. Now consider the sequence $\{x_n - 1\}$. By a study of the complements of the E_n sets—the sets on which $x_n - 1 = -1$ —it is readily seen that $x_n - 1 \rightarrow_H \theta$. By Theorem 1.3 it follows that $x_n \rightarrow_H 1$. Hence the sequence $\{x_n\}$ has the H -limits θ and 1.

Consider the sequence $\{x_n\}$ just defined together with the sequence $\{x'_n\}$, where $x'_n = 1 - x_n$. Since $\{x'_n\}$ has the H -limits θ and 1 it follows that $H\text{-}\lim_{n \rightarrow \infty} x_n + H\text{-}\lim_{n \rightarrow \infty} x'_n$ may equal either θ or 1. But $H\text{-}\lim_{n \rightarrow \infty} (x_n + x'_n) = 1$, so in an H -normed space $x_n \rightarrow_H x$ and $y_n \rightarrow_H y$ do not imply $(x_n + y_n) \rightarrow_H x + y$.

⁵ The r_n functions are closely related to the well known Rademacher functions.

REMARK. 3.2. By giving different interpretations to $\| (x_1, \dots, x_n) \|$ in the set of all functions defined on $(0, 1)$ and whose p th powers are summable the H -limit notion will be found to have different meanings. For example, in the case of all summable functions defined on $(0, 1)$ let $\| (x_1, \dots, x_n) \| = \max_{0 \leq t \leq 1} \inf \left(\left| \int_0^t x_1(t) dt \right|, \dots, \left| \int_0^t x_n(t) dt \right| \right)$. The set of all summable functions on $(0, 1)$ is an H -normed space according to this norm and it will be designated by $(LH)_1$. Since $\int_0^t x(t) dt$ is a continuous function of t on $(0, 1)$ it is evident from Theorem 2.1 that $x_n \rightarrow_H x$ is now equivalent to $\lim_{n \rightarrow \infty} \int_0^t x_n(t) dt = \int_0^t x(t) dt$ for every $t \in (0, 1)$.

4. The space (cH) . Consider the set of all convergent sequences $\{\xi_i\}$ of real numbers. Let an element x_n of the set be designated by $\{\xi_{ni}\}$ and let $\| (x_1, \dots, x_n) \| = \sup_{1 \leq i \leq \infty} |a_i^n|$, where by "sup" is meant the least upper bound of the numbers involved, and $a_i^n = \inf (|\xi_{1i}|, \dots, |\xi_{ni}|)$. Then the set is an H -normed space and it will be designated by (cH) .

THEOREM 4.1. A necessary and sufficient condition that $x_n \rightarrow_H x$ in (cH) is that

$$(a) \quad \lim_{n \rightarrow \infty} \xi_{ni} = \xi_i \quad \text{for } i = 1, 2, \dots,$$

$$(b) \quad \lim_{n \rightarrow \infty} \lim_{i \rightarrow \infty} \xi_{ni} = \lim_{i \rightarrow \infty} \xi_i.$$

PROOF. This theorem can be derived from Theorem 2.1 since (cH) can be put into a 1-1 norm preserving correspondence with a subset of (CH) . The correspondence is $\{\xi_i\}$ to $f(t)$ where $f(1/i) = \xi_i$, $f(0) = \lim_{i \rightarrow \infty} \xi_i$ and $f(t)$ is linear elsewhere.⁶

5. The space $(l^p H)$, $p \geq 1$. Consider the set of all sequences $\{\xi_i\}$ of real numbers such that $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$. Let any element x_n be designated by $\{\xi_{ni}\}$, and let $\| (x_1, \dots, x_n) \| = [\sum_{i=1}^{\infty} |a_i^n|^p]^{1/p}$, where $a_i^n = \inf (|\xi_{1i}|, \dots, |\xi_{ni}|)$. Then the set is an H -normed space and it will be designated by $(l^p H)$, $p \geq 1$.

THEOREM 5.1. A necessary and sufficient condition that $x_n \rightarrow_H x$ in $(l^p H)$ is that $\lim_{n \rightarrow \infty} \xi_{ni} = \xi_i$ for $i = 1, 2, \dots$.

PROOF. This theorem can be derived from statement (1) of Theorem 3.1 since $(l^p H)$, $p \geq 1$, can be put into a 1-1 norm preserving correspondence with a subset of $(L^p H)$, $p \geq 1$. The correspondence is $\{\xi_i\}$ to $f(t)$, where $f(t) = i(i+1)\xi_i$ for $1/(1+i) < t < 1/i$ and $f(t) = 0$ elsewhere.

⁶ The proofs of Theorems 4.1 and 5.1 were suggested by the referee. They are used here because they are much simpler than those originally given by the author.

6. Weak sequential convergence. In the last four sections certain concrete sets of elements have been considered as H -normed spaces by giving definite meanings to the norms of finite subsets of elements. Each of these sets may also be considered as a Banach space by proper definition of the norm for single elements. Banach has given necessary and sufficient conditions for weak sequential convergence⁷ in each case considered (Banach, pp. 134-137). An examination of our Theorems 2.1, 3.1, 4.1, and 5.1 will readily show that the necessary and sufficient conditions for $x_n \rightarrow_H x$ are in each case weaker than the necessary and sufficient conditions for weak sequential convergence. That is to say, $x_n \rightarrow_H x$ for the sets considered has been made weaker than weak sequential convergence by the interpretations given the norms of finite complexes. In this section different interpretations of the norms of finite complexes on the same sets considered in the previous sections will be given, and the resulting H -convergence notions will be found to be equivalent to weak sequential convergence.

To this end it is advisable to prove first a general theorem on Banach spaces and then to give appropriate interpretations to the norms of finite complexes in the different cases to be considered. Let E be a Banach space and let $\bar{E}$ be its conjugate, or the space of all linear limited functionals defined on E (Banach, p. 54). If $\{f_n\}$ is a sequence of and f is a single linear limited functional defined on E , then $f_n \rightarrow f$ will always be taken to mean $f_n(x) \rightarrow f(x)$ for every $x \in E$; i.e., $f_n \rightarrow f$ will mean that $\{f_n\}$ converges weakly to f (Banach, p. 122). Bounded sets of $\bar{E}$ will be weakly compact if every sequence chosen from a bounded (in the norm sense) set of linear limited functionals defined on E is such that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and another linear limited functional f_0 defined on E such that $f_{n_k} \rightarrow f_0$. It should be noted that if $f_n \rightarrow f$ and $\|f\| \leq M$, where M is independent of n , then $\|f\| \leq \lim_{n \rightarrow \infty} \|f_n\| \leq M$ (Banach, p. 123).

In the Banach space E let $\| (x_1, \dots, x_n) \| = \max_{\|f\|=1} G_n(f)$, where f is a linear limited functional on E and where $G_n(f) = \inf (|f(x_1)|, \dots, |f(x_n)|)$. $G_n(f)$ is a real valued function defined on $\bar{E}$. Because of the introduction of this norm for a finite complex the elements which form the Banach space E also form an H -normed space (EH). In this section the notions of H -convergence will be in terms of this norm.

LEMMA 6.1. *If $f_s \rightarrow f$ then $G_n(f_s) \rightarrow G_n(f)$.*

PROOF. This lemma follows readily from the fact that each $G_n(f)$ involves only a finite number of elements of E and from the definition of $f_s \rightarrow f$.

Let F be the subset of E such that if $f \in F$ then $\|f\| = 1$. Let F_0 be the closed extension of F in the sense that if $f_n \in F$ and $f_n \rightarrow f$ then $f \in F_0$.

LEMMA 6.2. *If the Banach space E is such that in $\bar{E}$ bounded sets are weakly compact, then $\max_{\|f\|=1} G_n(f)$ is attained by some element $f_0 \in F_0$.*

⁷ The sequence $\{x_n\}$ of a space X is said to be convergent to an element $x \in X$ in the weak sequential manner if for every linear (homogeneous and additive) and continuous functional $f(x)$ defined on X , $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ (Banach, p. 133).

PROOF. Consider any $G_{n_0}(f)$ and a sequence $\{f_s\}$ of elements of F , and let $\lim_{s \rightarrow \infty} G_{n_0}(f_s) = \max_{||f||=1} G_{n_0}(f)$. Each $G_{n_0}(f_s)$ is actually equal to the absolute value of f_s at some point of E ; for by definition each $G_{n_0}(f_s)$ involves only a finite number of elements of E . Therefore, $G_{n_0}(f_s) = |f_s(x_i)|$, where i assumes one of the values $1, 2, \dots, n_0$. Since the sequence $\{f_s\}$ is infinite there is some one x_{i_0} and an infinite subsequence $\{f_{s_p}\}$ of $\{f_s\}$ such that $G_{n_0}(f_{s_p}) = |f_{s_p}(x_{i_0})|$ for each integral p . Since each $||f_{s_p}|| = 1$ there is a subsequence $\{f_{s_{p_j}}\}$ of $\{f_{s_p}\}$ and an f_0 , of unit norm or less, such that $f_{s_{p_j}} \rightarrow f_0$. By Lemma 6.1 it follows that $\lim_{j \rightarrow \infty} G_{n_0}(f_{s_{p_j}}) = G_{n_0}(f_0)$. This completes the proof.

THEOREM 6.1. *If E is a Banach space such that in $\bar{E}$ bounded sets are weakly compact then $x_n \rightarrow_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x .*

PROOF. This theorem is a consequence of Lemmas 6.1 and 6.2 and its proof follows almost word by word the proof of Theorem 2.1.

It is well-known that the sets of elements which form the spaces (CH) , $(L^p H)$, $(l^p H)$, ($p \geq 1$), and (cH) are also elements of separable Banach spaces if the norm for single elements is properly chosen. Banach states (Banach, p. 123) that each separable Banach space E has a conjugate space whose bounded sets are weakly compact as sets of functionals over E . Consequently, if the norm of a finite complex is defined as in this section for any one of the concrete Banach spaces just mentioned, then $x_n \rightarrow_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x .

7. Comparison of limit notions. It is of interest to note how the gap between the H -convergence in H -normed spaces and the K -convergence in K -normed spaces may be bridged. It is first necessary to introduce a new limit notion.

DEFINITION 1⁰. A sequence $\{x_n\}$ of elements of an H -normed set X will be said to be convergent in the G -sense, or to be G -convergent, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that for $k \geq K_0$ there exists a $P_0 = P(e, k, \{x_{n_k}\})$ such that $k \geq K_0$, $s \geq K_0$ and $p \geq P_0$ imply

$$(1^0) \quad ||(x_{n_k} - x_{n_s}, \dots, x_{n_{k+p}} - x_{n_s})|| < e.$$

DEFINITION 2⁰. A sequence $\{x_n\}$ of elements of an H -normed set X will be said to be convergent to the element $x \in X$ in the G -sense, or to have x as its G -limit, written $x_n \rightarrow_G x$ or $G\text{-}\lim_{n \rightarrow \infty} x_n = x$, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that for $k \geq K_0$ there exists a $P_0 = P(e, k, \{x_{n_k}\})$ such that $k \geq K_0$ and $p \geq P_0$ imply

$$(2^0) \quad ||(x_{n_k} - x, \dots, x_{n_{k+p}} - x)|| < e.$$

THEOREM 7.1. *In a K -normed set $x_n \rightarrow_K x$ is equivalent to $x_n \rightarrow_G x$.*

PROOF. If $x_n \rightarrow_G x$ then given any $e > 0$ there exists an $N(e)$ and if $n \geq N(e)$ there exists a $P(e, n)$ such that $n \geq N(e)$ and $p \geq P(e, n)$ imply

$\|(x_n - x, \dots, x_{n+p} - x)\| < e$. But for all $0 \leq i \leq p$, $\|(x_n - x, \dots, x_{n+i} - x)\| \leq \|(x_n - x, \dots, x_{n+p} - x)\| < e$, because of Axiom *D* of *K*-normed sets. Hence, the required inequality holds for every $i \geq 0$, and so $x_n \rightarrow_K x$.

To prove the converse note that $x_n \rightarrow_K x$ implies $x_{n_i} \rightarrow_K x$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. From this fact it is immediately evident that $x_n \rightarrow_K x$ implies $x_n \rightarrow_G x$.

THEOREM 7.2. *If in a *K*-normed set a sequence $\{x_n\}$ is *K*-convergent then it is *G*-convergent, and conversely.*

In the proofs of Theorems 2.1, 3.1, 4.1, 5.1, and 6.1 we made use of a property which when generalized to cover all cases may be written as

PROPERTY 1. $\|(x_1, \dots, x_n, x_{n+1})\| \leq \|(x_1, \dots, x_n)\|$.

This property is actually Axiom *D* of *K*-normed sets with the inequality sign reversed. Since the norms of the spaces (CH) , (L^pH) , (l^pH) , ($p \geq 1$), (cH) and (EH) satisfy Property 1 it is evident that they are not *K*-normed spaces.

THEOREM 7.3. *In an *H*-normed space whose norm satisfies Property 1 $x_n \rightarrow_H x$ is equivalent to $x_n \rightarrow_G x$.*

PROOF. The proof of this theorem depends on Property 1 in much the same manner that the proof of Theorem depends on Axiom *D*.

THEOREM 7.4. *In an *H*-normed space whose norm satisfies Property 1 if a sequence is *G*-convergent then it is *H*-convergent, and conversely.*

REMARK 7.1. Consequently, the *G*-convergence notions may be used in place of the *H*-convergence notions and the *K*-convergence notions in the mutually exclusive classes of *H*-normed sets satisfying Property 1 and of *K*-normed sets, respectively.

8. *H-normed spaces.** By an *H**-normed space will be meant an *H*-normed space whose elements also form a linear normed space and in which convergence of a sequence $\{x_n\}$ to x in the norm sense (Banach convergence), i.e., $\|x_n - x\| \rightarrow 0$ —written $x_n \rightarrow_B x$ —implies $x_n \rightarrow_H x$. If an *H*-normed space satisfies Property 1 and if it is linear normed with respect to the norm of a finite complex when that norm operates on single elements then it is evidently an *H**-normed space. The *K*-normed spaces are not necessarily *H*-normed spaces (Vulich 1, p. 163) for the *K*-limit notion is in general stronger than the limit notion in the norm sense.

REMARK 8.1. The example of Remark 3.2 is an *H*-normed space satisfying Property 1. Moreover, the set of elements of that example form a linear normed space when the norm $\|x\| = \max_{0 \leq t \leq 1} \left| \int_0^t x(u) du \right|$ is used, and so it is an *H**-normed space. However, this is not the usual norm employed to make the set of summable functions on $(0, 1)$ a linear normed space—the usual norm is $\|x\| = \int_0^1 |x(u)| du$. Nevertheless, it is evident that the space $(LH)_1$ is an *H**-normed space when $x_n \rightarrow_B x$ is in accordance with the second norm just given and when $x_n \rightarrow_H x$ is with respect to the norm of a finite complex of $(LH)_1$.

By an $\bar{H}$ -normed space will be meant an H^* -normed space in which the H -limit is unique and satisfies the condition: $x_n \rightarrow_H x$ and $y_n \rightarrow_H y$ imply $(x_n + y_n) \rightarrow_H (x + y)$.

REMARK 8.2. The space $(L^p H)$, $p \geq 1$, is an H^* -normed space which is not $\bar{H}$ -normed.

LEMMA 8.1. If $\{a_n\}$ is a sequence of real numbers such that $a_n \rightarrow_B a$ and if x is an element of an H -normed space whose norm satisfies Property 1, then $a_n x \rightarrow_H ax$.

THEOREM 8.1. If $\{x_n\}$ is a sequence of elements of an $\bar{H}$ -normed space X whose norm satisfies Property 1, if $x_n \rightarrow_H x$ where $x \in X$, and if $\{a_n\}$ is a sequence of real numbers such that $a_n \rightarrow_B a$, then $a_n x_n \rightarrow_H ax$.

PROOF. $a_n x_n - ax = (a_n x_n - a_n x) + (a_n x - ax)$. From Theorem 1.5 it follows that the first expression on the right approaches θ in the H -sense, and from Lemma 8.1 it follows that the second expression does likewise. The theorem follows from the fact that the space is $\bar{H}$ -normed.

REMARK 8.3. An $\bar{H}$ -normed space whose norm satisfies Property 1 does not have the property that the derived class is closed; that is to say, if $x_n^m \rightarrow_H x_n$ for $n = 1, 2, \dots$, and if $x_n \rightarrow_H x$, it does not follow that there exists a subsequence $\{x_n^{m_n}\}$, where $m_1 < m_2 < \dots < m_n < \dots$, such that $x_n^{m_n} \rightarrow_H x$. For consider the following example from the space of all continuous functions defined on $(0, 1)$ with the norm defined as in §6. It has been shown in §6 that in this space $x_n \rightarrow_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x ; i.e., $x_n \rightarrow_H x$ is equivalent to $x_n(t) \rightarrow x(t)$ for each $t \in (0, 1)$ and to the existence of a constant $M > 0$ such that $\max_{0 \leq t \leq 1} |x_n(t)| \leq M$ (Banach, p. 134). Let $\{x_n\}$ be defined as follows:

$$\begin{aligned} x_n(t) &= x_n(0) = 0 \text{ for } 2/n \leq t \leq 1, \\ x_n(1/n) &= 1, \end{aligned}$$

$x_n(t)$ linear elsewhere. This sequence obviously has θ as its weak sequential limit. Let x_n^m be such that

$$\begin{aligned} x_n^m(t) &= x_n(t) \text{ for } 0 \leq t \leq 2/n, \\ x_n^m(t) &= 0 \text{ for } 2/n + 2/m \leq t \leq 1, \\ x_n^m(2/n + 1/m) &= n, \end{aligned}$$

$x_n^m(t)$ linear elsewhere. For each fixed value of n , $x_n^m \rightarrow x_n$ in the weak sequential sense. But since weak sequential convergence implies boundedness of the norms of elements it follows that it is impossible to find a sequence of integers $m_1 < m_2 < \dots < m_n < \dots$ such that $x_n^{m_n} \rightarrow_H \theta$.

9. Linear operations. Let X and Y be two sets of elements. If to each $x \in X$ there is made to correspond an element $y \in Y$ then an operation $y = U(x)$ is defined. X will be called the domain and Y the range of the operation.

DEFINITION 3. If $y = U(x)$ is an operation which transforms X into all or a part of Y , then if

- (1) X and Y are H -normed spaces $y = U(x)$ is said to be HH -continuous at the point x_0 if $x_n \rightarrow_H x_0$ implies $U(x_n) \rightarrow_H U(x_0)$,
 (2) X is an H -normed space and Y is a linear normed space $y = U(x)$ is said to be HB -continuous at the point x_0 if $x_n \rightarrow_H x_0$ implies $U(x_n) \rightarrow_B U(x_0)$,
 (3) X is a linear normed space and Y is an H -normed space $y = U(x)$ is said to be BH -continuous at the point x_0 if $x_n \rightarrow_B x_0$ implies $U(x_n) \rightarrow_H U(x_0)$,
 (4) X and Y are linear normed spaces $y = U(x)$ is said to be BB -continuous at the point x_0 if $x_n \rightarrow_B x_0$ implies $U(x_n) \rightarrow_B U(x_0)$.

If in any one of the above cases the specified continuity condition holds for every point of X then the operation is said to be continuous in that sense on X .

THEOREM 9.1. *If X is an H^* -normed space, Y is an H -normed space, and $y = U(x)$ is HH -continuous, then it is BH -continuous.*

PROOF. $x_n \rightarrow_B x$ implies $x_n \rightarrow_H x$ in an H^* -normed space, so by the HH -continuity of $y = U(x)$, $x_n \rightarrow_B x$ implies $U(x_n) \rightarrow_H U(x)$.

THEOREM 9.2. *If X is a linear normed space, Y is an H^* -normed space, and $y = U(x)$ is BB -continuous, then it is BH -continuous.*

THEOREM 9.3. *If X is an H -normed space, Y is an H^* -normed space, and $y = U(x)$ is HB -continuous then it is HH -continuous.*

THEOREM 9.4. *If the additive operation $y = U(x)$ transforms an H -normed space X into a space Y of the same type and is HH -continuous at a single point, then it is HH -continuous on X .*

THEOREM 9.5. *Statements similar to that of the previous theorem hold when $y = U(x)$ is HB -, BH -, or BB -continuous at a single point, X is H -normed, linear normed, or linear normed, respectively, and Y is linear normed, H -normed, or linear normed, respectively.*

THEOREM 9.6. *If $y = U(x)$ is an HH -continuous and additive operation from an $\bar{H}$ -normed space X to an $\bar{H}$ -normed space Y , then it is homogeneous.*

PROOF. The proof of this theorem depends on the uniqueness of the H -limit notion in $\bar{H}$ -normed spaces.

THEOREM 9.7. *Statements similar to that of the previous theorem hold when $y = U(x)$ is additive and HB -, BH -, or BB -continuous, X is $\bar{H}$ -normed, linear normed, or linear normed, respectively, and Y is linear normed, $\bar{H}$ -normed, or linear normed, respectively.*

In case the operation is from a domain X to the set of all real numbers, which set is readily seen to be an $\bar{H}$ -normed space when the norm of §1 is used, it will be called a functional and will be designated by $y = f(x)$. Since in the $\bar{H}$ -normed space of real numbers (see §1) the H - and B -convergence notions are equivalent (see Theorem 1.1), the four types of operations of Definition 3 reduce to two in the case of functionals. These will be referred to as the H - and B -continuous functionals.

10. General forms of H -continuous and linear functionals. Remark 10.1. Let X be an H^* -normed space. By Theorem 9.1 every H -continuous functional

on X is B -continuous on X . Consequently, the general form of the H -continuous and linear functional on X must be of the same nature as but less general than the general form of the B -continuous and linear functional on X .

10.1. The space (cH) . The elements of this space form a Banach space (c) when the norm of a finite complex operates as a norm for single elements. Then (cH) is an H^* -normed space. The general form of the B -continuous and linear functional for the space (c) is $f(x) = C \lim_{i \rightarrow \infty} \xi_i + \sum_{i=1}^{\infty} C_i \xi_i$ (Banach, p. 66).

THEOREM 10.1. *The general form of the H -continuous and linear functional on (cH) is $f(x) = C \lim_{i \rightarrow \infty} \xi_i + \sum_{i=1}^n C_i \xi_i$.*

PROOF. By Remark 10.1 every H -continuous and linear functional on (cH) is of the form of the B -continuous and linear functional on the same elements taken as a Banach space but with additional restrictions on the constant C_i 's. Assume that $f(x)$ is H -continuous and linear and that there is an infinite number of non-zero constant C_i 's involved in its representation. Consider the sequence $\{x_n\}$ of (cH) where $\xi_{ni} = 0$ if $n \neq i$ and $\xi_{nn} = \text{sgn } 1/C_n$. This sequence is obviously such that $x_n \rightarrow_H \theta$. But for an infinite number of values of n , $f(x_n) = 1$, so $\lim_{n \rightarrow \infty} f(x_n) = 1$. Consequently, $x_n \rightarrow_H \theta$ does not imply $f(x_n) \rightarrow_B 0$. Therefore, the possibility of the general form of the H -continuous and linear functional on (cH) involving more than a finite number of non-zero constants has been eliminated. Consequently, we are confined to a consideration of the form $C \lim_{i \rightarrow \infty} \xi_i + \sum_{i=1}^n C_i \xi_i$. This functional is readily seen to be H -continuous and linear.

10.2. The space $(l^p H)$, $p \geq 1$. Remarks similar to those preceding Theorem 10.1 apply here.

THEOREM 10.2. *The general form of the H -continuous and linear functional on $(l^p H)$, $p \geq 1$, is $f(x) = \sum_{i=1}^n C_i \xi_i$.*

PROOF. The proof of this theorem is similar to the proof of Theorem 10.1.

REMARK 10.2. If K is a finite complex then by $U(K)$ is meant the complex of the images of the elements of K by $y = U(x)$ (Vulich 1, p. 165). Vulich has proved the following theorem for operations defined on K -formed spaces: A necessary and sufficient condition that the additive operation $y = U(x)$ be KK -continuous is that there exist a constant $C > 0$ such that for every complex K of the domain $\|U(K)\| \leq C \|K\|$. A similar statement cannot be made for HH -continuous and additive operations on H^* -normed spaces. For in the space $(l^p H)$, $p \geq 1$, let K be the complex (x_1, x_2) where $\xi_{pi} = 0$ if $p \neq i$ and $\xi_{pp} = 1$, and let $f(x) = \xi_1 + \xi_2$. Then $\|(x_1, x_2)\| = 0$ and $\|f(x_1), f(x_2)\| = 1$. Hence, there exists no constant $C > 0$ such that $\|f(K)\| \leq C \|K\|$.

10.3. The space (CH) . The elements of (CH) form a Banach space (C) when the norm of a finite complex operates on single elements only. Then (CH) is an H^* -normed space. The general form of the B -continuous and linear functional defined on the elements of (CH) when they are taken as the Banach space

(C) is $f(x) = \int_0^1 x(t) dg(t)$, where $g(t)$ is a function of bounded variation and $x(t)$ is of (C) (Banach, p. 61).

THEOREM 10.3. *The general form of the H -continuous and linear functional on (CH) is $f(x) = \int_0^1 x(t) dg(t)$, where $g(t)$ is a function of bounded variation which is constant except for a finite number of discontinuities. An alternate form is $f(x) = \sum_{i=1}^n C_i x(t_i)$.*

PROOF. It will be shown first that $g(t)$, which must be of bounded variation because of Remark 10.1, cannot be continuous and non-decreasing (or non-increasing) on $(0, 1)$ with $g(t) > g(0)$ (or $g(t) < g(0)$) for $t > 0$. Suppose that $g(t)$ is of this form. Define a sequence $\{x_n\}$ of continuous functions as follows:

$$x_n(t) = x_n(0) = 0 \text{ for } 2/n \leq t \leq 1,$$

$$x_n(1/n) = I^{-1}, \text{ where } I = \int_0^{1/n} t dg(t),^8$$

and $x_n(t)$ linear elsewhere. Obviously, $x_n \rightarrow_H \theta$. It will be shown that $\lim_{n \rightarrow \infty} \int_0^1 x_n(t) dg > 0$.

Write $\int_0^1 x_n(t) dg = \int_0^{1/n} x_n(t) dg + \int_{1/n}^{2/n} x_n(t) dg$. On the interval $(0, 1/n)$, $x_n(t) = ntI^{-1}$. Then $\int_0^{1/n} x_n(t) dg = \int_0^{1/n} nI^{-1}t dg = nI^{-1}I = n$. On the other hand, on the interval $(1/n, 2/n)$, $x_n(t) = -nI^{-1}(t - 2/n)$, whence $\int_{1/n}^{2/n} x_n(t) dg = -nI^{-1} \int_{1/n}^{2/n} (t - 2/n) dg = -nI^{-1} \int_{1/n}^{2/n} t dg + 2I^{-1} \int_{1/n}^{2/n} dg$. Since in $(1/n, 2/n)$, $t \leq 2/n$, it follows that $\int_{1/n}^{2/n} t dg \leq 2/n \int_{1/n}^{2/n} dg$, and so $\int_{1/n}^{2/n} x_n(t) dg \geq -2I^{-1} \int_{1/n}^{2/n} dg + 2I^{-1} \int_{1/n}^{2/n} dg = 0$. Consequently, $\int_0^1 x_n(t) dg \geq n$, and therefore, if $g(t)$ is of the form specified at the beginning

of this proof, $x(t)$ is of (CH), and $f(x) = \int_0^1 x(t) dg(t)$, then there is a sequence $\{x_n\}$ such that $x_n \rightarrow_H \theta$ but for which $\{f(x_n)\}$ does not approach zero as $n \rightarrow \infty$. Then $f(x)$ is not H -continuous. So, in general if $f(x)$ is H -continuous and linear it may not be of the form $\int_0^1 x(t) dg(t)$, where $g(t)$ is continuous and such that for some interval (t_0, t') , $g(t_0) < g(t)$ (or $g(t_0) > g(t)$) for $t > t_0$. Since if this were the case the sequence $\{x_n\}$ used above could be constructed on the interval (t_0, t') and then a contradiction would follow.

Therefore, $g(t)$ must be constant except for at most a denumerable number of points of discontinuity of the first kind. Assume that $g(t)$ is of this nature

⁸ By the integration by parts theorem for Stieltjes integrals it is readily shown that the denominator does not vanish.

and that $f(x)$ is an H -continuous and linear functional on (CH) . Let $\{a_n\}$ be a sequence of points of $(0, 1)$ which approaches some a_0 from the right (or left) and which has the property that at each a_n the function $g(t)$ has a discontinuity of measure $b_n \neq 0$. Each a_n may itself be the limit point of points of discontinuity of $g(t)$. Let such points be designated by a_{np} and let the measure of the discontinuity of $g(t)$ at a_{np} be $b_{np} \neq 0$. About each a_n choose an interval I_n such that the left-hand end point of I_n is to the right of a_0 , such that the length of $I_n \leq \min(\text{length } I_{n-1}, 1/n)$, and such that if $a_{np} \in I_n$ then $\sum_{p=1}^{\infty} |b_{np}| \leq |b_n|/2^n$. Define a sequence of continuous functions as follows: let $x_n(a_n) = \text{sgn } 1/b_n$; on the complement of I_n let $x_n(t) = 0$, and finally let $x_n(t)$ be linear elsewhere. Then $x_n \rightarrow_H \theta$.
$$\int_0^1 x_n(t) dg = \sum_{p=1}^{\infty} [g(a_{np} + 0) - g(a_{np} - 0)]x_n(a_{np}) + b_n x_n(a_n) = \sum_{p=1}^{\infty} b_{np} x_n(a_{np}) + 1.$$
 If each $b_{np} x_n(a_{np}) < 0$, then
$$\int_0^1 x_n(t) dg = 1 - \sum_{p=1}^{\infty} |b_{np}| |x_n(a_{np})| \geq 1 - \sum_{p=1}^{\infty} |b_{np}|/2^n |b_n| > 1/2.$$
 If each $b_{np} x_n(a_{np})$ is not less than zero then $\int_0^1 x_n(t) dg$ is obviously greater than $1/2$. Therefore, although $x_n \rightarrow_H \theta$ it does not follow that $f(x_n) \rightarrow_B 0$. Hence, if $f(x)$ is to be H -continuous and linear it is impossible that $g(t)$ be other than constant except for at most a finite number of points of discontinuity of the first kind. In this case $f(x) = \int_0^1 x(t) dg = \sum_{i=1}^n x(t_i)[g(t_i + 0) - g(t_i - 0)] = \sum_{i=1}^n C_i x(t_i)$. Since only a finite number of terms are involved in this form $f(x)$ is obviously H -continuous and linear.

REMARK 10.3. It is evident from this theorem that there are B -continuous and linear functionals which are not H -continuous. Therefore, the converse of Theorem 9.1 is not true.

10.4. The space $(L^p H)$, $p \geq 1$. THEOREM 10.4. If $y = U(x)$ is a continuous transformation from $(L^p H)$, $p \geq 1$, to a space Y in which the limit notion is unique, then $y = U(x)$ is a constant.⁹

PROOF. Let $x \in (L^p H)$, $p \geq 1$, and let $y_n(t) = x(t) \cdot 1/2(1 + r(t))$, where $r_n(t)$ is the function defined in Remark 3.1. As in Remark 3.1 it is seen that $y_n \rightarrow_H \theta$ and $y_n \rightarrow_H x$. Since $y = U(x)$ is continuous it follows that $U(y_n) \rightarrow U(\theta)$ and $U(y_n) \rightarrow U(x)$. Because of the uniqueness of the limit notion in Y , $U(x) = U(\theta)$.

COROLLARY 10.41. If $y = f(x)$ is an H -continuous and linear functional from $(L^p H)$, $p \geq 1$, then $f(x) \equiv 0$.

11. General forms of different types of linear operations. If for the set of all continuous functions defined on $(0, 1)$ the norm for single elements is taken to be $\max_{0 \leq t \leq 1} |x(t)|$, then the set is a Banach space and it will be denoted by (C) .

With respect to this norm the space (CH) is an H^* -normed space. It has been

⁹ This theorem and its proof were supplied by the referee. It is a generalization of a theorem originally given by the author.

shown (Fichtenholz, p. 32) that the general form of the *BB*-continuous and linear operation from (C) to (C) is given by

$$(1) \quad y(s) = U(x) = \int_0^1 x(t) d_t K(s, t),$$

where $K(s, t)$ is a function of two variables defined on $0 \leq s, t \leq 1$ and is such that

(1⁰) $K(s, t)$ is of bounded variation as a function of t for each s and $\text{var}_{0 \leq t \leq 1} K(s, t) \leq C$, where C is a positive constant independent of s ,

(2⁰) $K(s, 0) = 0$; $K(s, 1)$ is continuous in s ,

(3⁰) $K(s, t)$ is continuous in measure¹⁰ with respect to s in the interval $(0, 1)$ for each $s = s_0$.

If $y(s) = U(x)$ is *BH*-continuous and linear from (CH) to (CH) , then $y(s_0) = U(x)$ is a *B*-continuous and linear functional defined on (CH) and it may be represented by $y(s_0) = U(x) = \int_0^1 x(t) d_t K(s_0, t)$, and so $y(s) = U(x)$ is of form

(1) with restrictions on $K(s, t)$ which will now be examined.

THEOREM 11.1. *The class of all BH-continuous and linear operations from (CH) to (CH) is equivalent to the class of all BB-continuous and linear operations from (CH) to (CH) .*

PROOF. Because of Theorem 9.2 it is evident that the class of all *BB*-continuous and linear operations from (CH) to (CH) is a subclass of the class of all *BH*-continuous and linear operations between the same spaces. Therefore, since each *BH*-continuous and linear operation is of the form (1), in order that an operation be *BH*-continuous and linear without being *BB*-continuous and linear it is necessary that some one of the conditions 1⁰, 2⁰, 3⁰ on $K(s, t)$ be weakened. But these conditions are necessary and sufficient not only for the *BB*-continuity and linearity of $y = U(x)$ but also for the continuity of $y(s) = U(x)$. Hence, if one of these conditions is weakened $y(s)$ will no longer be continuous. Therefore, every *BH*-continuous and linear operation must satisfy conditions 1⁰, 2⁰, and 3⁰ and our theorem is proved.

THEOREM 11.2. *The general form of the HH-continuous and linear operation from (CH) to (CH) is given by (1) where $K(s, t)$ satisfies conditions 1⁰, 2⁰, 3⁰ and also*

(4⁰) *for s constant $K(s, t)$ is constant in t except for at most a finite number of points of discontinuity of the first kind.*

PROOF. If $y(s) = U(x)$ is of form (1) and if $K(s, t)$ satisfies conditions 1⁰-4⁰, then $y(s)$ is continuous because of conditions 1⁰-3⁰. Moreover, if $x_n \rightarrow_H x$, then $y_n(s) \rightarrow y(s)$ for each $s \in (0, 1)$ and so $y_n \rightarrow_H y$. For consider any $s = s_0$, then $|y_n(s_0) - y(s_0)| \leq \left| \int_0^1 [x_n(t) - x(t)] d_t K(s_0, t) \right|$. The last expression is the

¹⁰ $K(s, t)$ is said to be convergent in measure with respect to s for $s = s_0$ on $0 \leq t \leq 1$ if $K(s, t)$ as a function of t approaches $K(s_0, t)$ in measure when $s \rightarrow s_0$.

absolute value of an H -continuous and linear functional (see Theorem 10.4 and condition 4⁰), and therefore, $x_n \rightarrow_H x$ implies $y_n(s_0) \rightarrow y(s_0)$. This proves our contention.

If $y(s) = U(x)$ is an HH -continuous and linear operation from (CH) to (CH) , then it is of form (1) and $K(s, t)$ satisfies 1⁰-4⁰. For consider any $s = s_0$, then $y(s_0) = U(x)$ is merely an H -continuous and linear functional on (CH) . Hence, $y(s_0) = \int_0^1 x(t) d_t K(s_0, t)$ and $K(s_0, t)$ must satisfy 4⁰ because of the restriction of Theorem 10.4. Then $y(s) = \int_0^1 x(t) d_t K(s, t)$. Since $y(s) = U(x)$ is HH -

continuous and linear and (CH) is an H^* -normed space, it follows from Theorem 9.1 that $y(s)$ is BH -continuous and linear. By Theorem 11.1, $y(s) = U(x)$ is also BB -continuous and linear and so $K(s, t)$ must satisfy conditions 1⁰-3⁰.

THEOREM 11.3. *The general form of the HB -continuous and linear operation from (CH) to (CH) is given by (1) where $K(s, t)$ satisfies conditions 1⁰-3⁰ and also (5⁰) there exists at most a finite number of points $t_1, t_2, \dots, t_q$ such that for any value $s = s_0$ the points of discontinuity of $K(s_0, t)$ are to be found among the points $t_i (i = 1, 2, \dots, q)$.*

PROOF. As in the proof of Theorem 11.2 it is readily seen that any HB -continuous and linear operation is of the form (1). If $y = U(x)$ is HB -continuous and linear, then by Theorem 9.3 it is HH -continuous and linear, and so by Theorem 11.2 $K(s, t)$ satisfies conditions 1⁰-3⁰. Suppose that 5⁰ is not satisfied by $K(s, t)$. Then there exists a sequence $\{s_n\}$ of points of the interval $(0, 1)$ such that the points of discontinuity of the first kind which are contributed by $\{K(s_n, t)\}$ are infinite in number, that is to say, there is an infinite sequence $\{t_v\}$ of points such that each is a point of discontinuity of some $K(s_n, t)$. Choose a subsequence $\{t_{v_p}\}$ of $\{t_v\}$ such that $\{t_{v_p}\}$ approaches some t_0 from the right (or left). About each t_{v_p} as center and associated with each $K(s_n, t)$ there is an interval $I(t_{v_p}, s_n)$ such that within it there is no point of discontinuity of $K(s_n, t)$ except perhaps t_{v_p} itself. The existence of this interval is assured by the HH -continuity of each HB -continuous operation. Let I_{v_p} be an interval $I(t_{v_p}, s_n)$ which does not contain t_0 and whose right-hand end point is to the left of $t_{v_p} + (t_{v_p} - t_0)/2$ and which is such that the measure $a_{v_p}^n$ of the discontinuity of $K(s_n, t)$ at t_{v_p} is not equal to zero. Define a sequence $\{x_n\}$ of continuous functions as follows:

$$\begin{aligned} x_n(t_{v_p}) &= \operatorname{sgn} 1/a_{v_p}^n, \\ x_n(t) &= 0 \text{ for } t \text{ not of } I_{v_p}, \end{aligned}$$

and $x_n(t)$ linear elsewhere. Then $x_n \rightarrow_H \theta$. Moreover, $y_n(s) = \int_0^1 x_n(t) d_t K(s, t)$ is such that $|y_n(s_n)| = |x_n(t_{v_p})| \cdot |K(s_n, t_{v_p} + 0) - K(s_n, t_{v_p} - 0)| = |\operatorname{sgn} 1/a_{v_p}^n| \cdot |a_{v_p}^n| = 1$. Therefore, $\|y_n\| = \max_{0 \leq s \leq 1} |y_n(s)| = 1$, and so $x_n \rightarrow_H \theta$ does not imply $y_n \rightarrow_B \theta$. This is a contradiction and so 5⁰ must hold.

Let $y(s) = U(x)$ be of the form (1) and be defined on (CH) , and let $K(s, t)$ satisfy conditions 1^0 , 2^0 , 3^0 , and 5^0 . Because of conditions 1^0 - 3^0 , $y(s)$ is continuous. Moreover, $y(s) = \int_0^1 x(t) d_t K(s, t) = \sum_{i=1}^q x(t_i)[K(s, t_i + 0) - K(s, t_i - 0)] = \sum_{i=1}^q x(t_i)C_i(s)$ by condition 5^0 . From 1^0 we know that there exists a constant $C > 0$ such that $|C_i(s)| \leq C$ for each i and for $s \in (0, 1)$. Then $|y_n(s) - y(s)| \leq C \sum_{i=1}^q |x_n(t_i) - x(t_i)|$. If $x_n \rightarrow_H x$, then since only a finite number of points are involved in the last summation, $\|y_n - y\| \rightarrow 0$.

PHILADELPHIA COLLEGE OF PHARMACY

BIBLIOGRAPHY

- S. Banach, *Théorie des Opérations Linéaires* (1932).
- C. Caratheodory, *Vorlesungen über Reelle Funktionen* (1918).
- G. Fichtenholz, *Sur les opérations linéaires dans l'espace des fonctions continues*, Bull. Acad. Roy. Belg. (1936), v. 22, pp. 26-33.
- B. Vulich 1, *On a generalized notion of convergence in a Banach space*, Annals of Math., v. 38, pp. 156-174.
- B. Vulich 2, *Some remarks to the theory of K -normed spaces*, Comptes Rendus de l'Académie des Sciences de l'U.R.S.S. (1936), v. 2, #2, pp. 59-62.
- B. Vulich 3, *Correction to the note "Some remarks to the theory of K -normed spaces,"* Comptes Rendus de l'Académie des Sciences de l'U.R.S.S. (1936), v. 3, #3, pp. 109-110.

